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1997 J. Phys. A: Math. Gen. 30 6951

(http://iopscience.iop.org/0305-4470/30/19/029)

J. Phys. A: Math. Gen. **30** (1997) 6951–6962. Printed in the UK PII: S0305-4470(97)81045-1

Complex analysis of a piece of Toda lattice

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Received 13 January 1997

Abstract. We study a small piece of two-dimensional Toda lattice as a complex dynamical system. In particular, it is shown analytically how the Julia set, which appears when the piece is deformed, disappears as the system approaches the integrable limit.

1. Introduction

The two-dimensional Toda lattice is one of the soliton equations which has become more and more important as a key object in theoretical physics. It was first formulated by Hirota in 1981 [1] as a discrete version of the two-dimensional continuous time Toda lattice and its equivalence to the KP hierarchy was shown by Miwa [2]. When quasi-periodic solutions are substituted, it is nothing but the identity known as Fay's trisecant formula which characterizes algebraic curves.

This equation has become known in other fields of physics in the last ten years. It was shown to be satisfied by the string amplitudes in particle physics [3]. More recently there have appeared papers demonstrating the unexpected correlation of this equation with other topics in physics. The transfer matrix of the solvable lattice model with A_l symmetry, for example, was shown to satisfy this equation [4, 5]. This equation has also been proven to unify discrete Painlevé equations [6]. The connection of solvable cellular automata to this equation offers another example [7].

Completely integrable nonlinear systems must play fundamental roles in various phenomena in physics. It is remarkable that many integrable systems in different fields are unified into a single equation. We are interested in clarifying the ultimate notion of integrability of the systems. Investigation of such systems themselves, however, will not reveal all features of the systems. The real meaning of integrability will be clarified only in comparison with non-integrable systems.

An arbitrary deformation of the two-dimensional Toda lattice will destroy integrability and create chaos. Since the system contains an infinite number of degrees of freedom it is extremely difficult to study analytically the behaviour of the transition from non-integrable to integrable phases. It should be recalled that very little is known about analytical properties of non-integrable systems. The main part of the studies of complex dynamical systems have been limited to simple systems with one degree of freedom.

Very recently we pointed out [8] that a set of lattice points, which form a parallelogram in the two-dimensional lattice space, constitute a piece of the Toda lattice. We call it a Toda molecule [9] since it is essentially what is intended by this name, but it has been used in a slightly different context in the literature. The remarkable fact is that the small pieces can be separated from other parts without losing any properties of the original Toda lattice.

The purpose of this paper is to study in detail analytical properties of the smallest piece of Toda molecules. The smallest Toda molecule is a smallest parallelogram of four lattice points. We will call it a Toda atom for convenience. Since every Toda molecule preserves properties possessed by the Toda lattice, we can study analytical properties of the system from the knowledge of a Toda atom. In the first part of this paper we show that the time evolution of a Toda atom is described by an iterative Möbius map. The form invariance of this map certificates integrability of this system. In the second part of this paper we will consider a deformation of this piece. Under generic deformation chaos will be generated through the time evolution. We are especially concerned with the analytical property of the Julia set as the system approaches to the integrable map. We will show how the Julia set converges to the points on the orbit of a Möbius map as a parameter, which interpolates between integrable and nonintegrable maps, approaches the critical limit.

2. Pieces of Toda lattice

In this section we show that the two-dimensional Toda lattice can be cut into small pieces without losing any properties possessed by the original system. To begin with, let us write down the equation which was derived by Hirota as a discrete version of the two-dimensional continuous time Toda lattice [1]:

$$\begin{aligned}&\alpha g_n(l+1,m)g_n(l,m+1)+\beta g_n(l,m)g_n(l+1,m+1)\\&\qquad -(\alpha+\beta)g_{n+1}(l+1,m)g_{n-1}(l,m+1)=0 \qquad \alpha,\beta\in\mathbb{C},\, l,m,n\in\mathbb{Z}. \quad (1)\end{aligned}$$

We called this equation the Hirota bilinear difference equation (HBDE)†. This is a nonlinear system defined on the three-dimensional lattice space. Our key observation is the following. For a fixed point of the lattice $(l,m,n)=(\bar{l},\bar{m},\bar{n})$, we denote by $\mathbb{A}$ the set of points $(\bar{l},\bar{m},\bar{n})$, $(\bar{l}+1,\bar{m},\bar{n})$, $(\bar{l},\bar{m}+1,\bar{n})$, $(\bar{l}+1,\bar{m}+1,\bar{n})$, $(\bar{l}+1,\bar{m},\bar{n}+1)$, $(\bar{l},\bar{m}+1,\bar{n}-1)$. Then if $g_n(l,m)$ is a solution of (1),

$$f(l,m,n)=\begin{cases} g_n(l,m) & (l,m,n)\in\mathbb{A}\\ 0 & \text{otherwise}\end{cases} \qquad (2)$$

is also a solution of (1). This is the smallest piece of the Toda lattice.

The proof is simple. Because $\mathbb{A}$ is surrounded by zero, every equation on other pieces is automatically satisfied. The result can be easily generalized to a larger parallelogram prism when it is surrounded by zero. We call it a Toda molecule according to [9]. Then it will be natural to call (2) a Toda atom. If there are many Toda molecules in the three-dimensional lattice space separated by zeros from each other it is again a solution of (1). A slice perpendicular to the l axis of such example is presented in figure 1.

For an illustration let us consider the one soliton state localized on the smallest parallelogram specified by $(m,n)=(0,0)$, $(0,1)$, $(1,-1)$, $(1,0)$ on the (m,n) lattice plane, but allowed to range all integers along l. Now we recall that in the usual lattice space the one soliton solution is given by [2, 10]

$$f^{1\mathrm{sol}}(l,m,n)=\prod_j(1-az_j)^{-k_j}+\prod_j(1-bz_j)^{-k_j}. \qquad (3)$$

Here a,b are arbitrary constants and $\{z_j\}$ are parameters which determine the velocity of the soliton. $\{k_j\}$ are variables taking values on integers. We can choose any three among

† This equation is also called the Hirota–Miwa equation in recent literature.

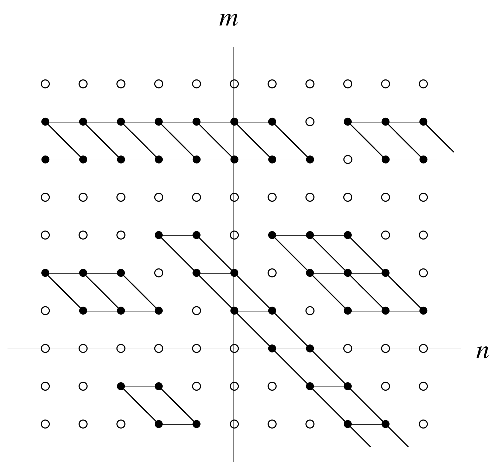

Figure 1.

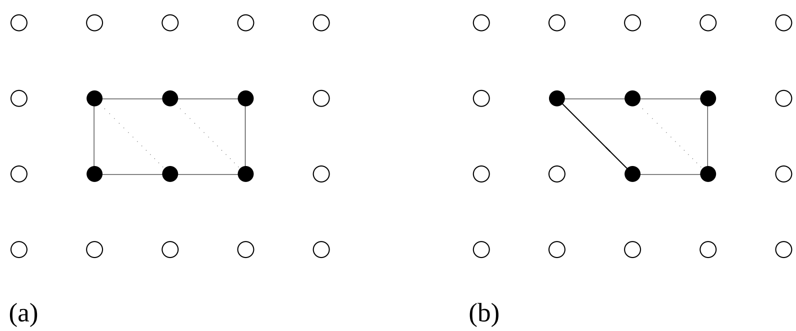

Figure 2.

$\{k_j\}$ to relate them to our variables (l, m, n). Let k_1, k_2, k_3 be such a three and relate them according to

$$k_1 = m + n - \tfrac{1}{2} \qquad k_2 = -m - \tfrac{1}{2} \qquad k_3 = l - n - \tfrac{1}{2}. \tag{4}$$

Writing (3) explicitly we find

$$\begin{aligned}
f^{1\mathrm{sol}}(l, 0, 0) &= A(1 - az_3)^{-l} + B(1 - bz_3)^{-l} \\
f^{1\mathrm{sol}}(l, 1, 0) &= A\frac{1 - az_2}{1 - az_1}(1 - az_3)^{-l} + B\frac{1 - bz_2}{1 - bz_1}(1 - bz_3)^{-l} \\
f^{1\mathrm{sol}}(l, 0, 1) &= A\frac{1 - az_3}{1 - az_1}(1 - az_3)^{-l} + B\frac{1 - bz_3}{1 - bz_1}(1 - bz_3)^{-l} \\
f^{1\mathrm{sol}}(l, 1, -1) &= A\frac{1 - az_2}{1 - az_3}(1 - az_3)^{-l} + B\frac{1 - bz_2}{1 - bz_3}(1 - bz_3)^{-l}
\end{aligned} \tag{5}$$

where

$$A := \sqrt{(1 - az_1)(1 - az_2)(1 - az_3)} \qquad B := \sqrt{(1 - bz_1)(1 - bz_2)(1 - bz_3)}.$$

We see from (5) that all points belonging to the same piece behave similarly. The parameters are related to α, β of (1) by

$$\alpha = z_1(z_2 - z_3) \qquad \beta = z_2(z_3 - z_1)$$

for (5) to satisfy the HBDE. If we define the amplitude $\varphi_{mn}(l)$ by

$$\varphi_{mn}(l) := \frac{f(l+1,m,n)f(l-1,m,n)}{f^2(l,m,n)} - 1 \tag{6}$$

it behaves as

$$\varphi_{00}^{1\mathrm{sol}}(l) = \frac{\sinh^2 p}{\cosh^2(pl+\chi)} \qquad p := \frac{1}{2}\ln\frac{1-az_3}{1-bz_3} \qquad \chi := \frac{1}{2}\ln\frac{B}{A}. \tag{7}$$

This represents a localized peak along the l axis. The other amplitudes $\varphi_{mn}(l)$ behave almost the same but different by the values of the phase χ.

If we consider an evolution of the system in variable l, a Toda atom is composed of four lattice points. Since there is only one equation of motion (1), they are not independent variables. Three of them can be chosen arbitrarily leaving one to be determined by the equation. Let z_l be $f(l,0,0)$. The other three could be either dependent or independent of z_l. If they are independent of z_l, the equation of motion is linear in z_l. On the other hand, if they do depend on z_l they are allowed to be at most linear in z_l, for the equation to remain Hirota bilinear form. Namely, we can write

$$f(l,m,n) = A_{m,n}z_l + B_{m,n} \qquad (m,n) = (0,0), (1,0), (0,1), (1,-1) \tag{8}$$

with $A_{0,0} = 1$, $B_{0,0} = 0$. Upon substituting them together into the HBDE, it is easy to see that we obtain an equation of the form

$$\begin{aligned} &z_{l+1} = \frac{Az_l + B}{Cz_l + D} \\ &A = -\beta B_{1,0} + (\alpha+\beta)B_{0,1}A_{1,-1} \qquad B = (\alpha+\beta)B_{0,1}B_{1,-1} \\ &C = (\alpha+\beta)(A_{1,0} - A_{0,1}A_{1,-1}) \qquad D = \alpha B_{1,0} - (\alpha+\beta)A_{0,1}B_{1,-1}. \end{aligned} \tag{9}$$

(9) is a Möbius map. Therefore, the map is integrable.

If we remember that the HBDE is invariant under the transformation of $f(l,m,n) \to \mathrm{e}^{al+bm+cn}f(l,m,n)$, the one soliton solution (5) offers an example of (8).

The solution of (9) can be obtained as follows. A Möbius map has three fixed points. By an appropriate transformation, $z_l \to \phi \circ z_l \circ \phi^{-1}$, one of the fixed points can be transformed into 0. After the transformation the map will have the form

$$z_{l+1} = \mu\frac{z_l}{1+\nu z_l}. \tag{10}$$

(10) is easily solved for an arbitrary initial value z_0 to get

$$z_l = \frac{\mu^l z_0}{1 + \nu(1-\mu^l/1-\mu)z_0}. \tag{11}$$

Applying to this the inverse transformation $z_l \to \phi^{-1} \circ z_l \circ \phi$, the general solution to (9) is obtained. We call the map (10) the integrable logistic map (ILM). The meaning of this name will become clear later.

We notice that (10) corresponds to the case in which one of the lattice points is fixed constant and the other three points behave the same:

$$\begin{aligned} &f^{\mathrm{ILM}}(l,0,0) = f^{\mathrm{ILM}}(l,0,1) = f^{\mathrm{ILM}}(l,1,-1) =: z_l \\ &f^{\mathrm{ILM}}(l,1,0) = \frac{\mu-1}{\nu} \qquad \mu = -\frac{\beta}{\alpha}. \end{aligned} \tag{12}$$

What does the amplitude look like in this case? To see we substitute (11) into (6) and get

$$\varphi^{\mathrm{ILM}} = \frac{\sinh^2 p}{\cosh^2(pl+\chi) - \cosh^2 p} \qquad p := \frac{1}{2}\ln\mu \qquad \chi := \frac{1}{2}\ln\frac{\mu z_0}{1-\mu+\nu z_0}. \tag{13}$$

The similarity of this result to the one soliton solution (7) must be apparent.

We may further simplify the equation by

$$f^{\mathrm{lin}}(l,0,0) =: z_l \qquad f^{\mathrm{lin}}(l,1,-1) = f^{\mathrm{lin}}(l,1,0) = 1-\frac{1}{\mu} \qquad f^{\mathrm{lin}}(l,0,1) = c \tag{14}$$

where c is a constant.

The map turns out to be linear

$$z_{l+1} = \mu z_l + (1-\mu)c \tag{15}$$

and yields the solution

$$z_l = \mu^l(z_0 - c) + c. \tag{16}$$

The corresponding amplitude is

$$\varphi^{\mathrm{lin}}(l) = \frac{\sinh^2 p}{\cosh^2(pl+\chi)} \qquad p = \frac{1}{2}\ln\mu \qquad \chi = \frac{1}{2}\ln\frac{z_0 - c}{c} \tag{17}$$

which is again the form of (7).

3. Generalized logistic map

As we have learned in the preceeding section, the smallest piece of Toda lattice already possesses useful information of the integrable dynamical systems. In this section we study a deformation of the Toda atom. There are many different forms of deformation, some of which preserve integrability and some which destroy it. Since we are interested in studying the transition between integrable and non-integrable maps, we must break the integrability of the Toda atom.

For this purpose we recall that the Toda molecules have a characteristic form as seen in figure 1. Their cross sections in the (m, n) plane are parallelograms declined to the same direction. This is due to the property of the Toda atom defined in (2). The very reason for this asymmetry comes from the asymmetry† of the HBDE under the exchange of l and m as seen in (1). The equation in which the role of l and m in (1) are exchanged is also integrable. In fact, we could start from this equation without changing any of the results.

From this argument we are tempted to consider the following deformation of the HBDE:

$$\begin{aligned}\alpha g_n(l+1,m)g_n(l,m+1) + \beta g_n(l,m)g_n(l+1,m+1) - (\alpha+\beta)[(1-\gamma)\delta g_{n+1}(l+1,m) \\ +\gamma g_{n+1}(l,m+1)] \times [(1-\gamma')\delta' g_{n-1}(l,m+1) + \gamma' g_{n-1}(l+1,m)] = 0.\end{aligned} \tag{18}$$

We note that this equation is integrable when $\gamma = \gamma' = 0$ and $\delta\delta' = 1$, or $\gamma = \gamma' = 1$. Integrability of other cases is not known at this point. Moreover, we are not able to separate some small part of the lattice independently from the rest as was done to get a Toda atom. Nevertheless, it is worth studying (18) defined on a portion of the lattice shown in figure 2(a).

† If we had chosen another set of variables, the HBDE would look more symmetric and the corresponding Toda atom could be either cubic or octahedron [11]. We have used asymmetric variables so that deformations can be discussed.

In order to proceed further we have to specify the model so that we can study analytical properties of the map explicitly. We will consider, in the following discussion, the map given by (10) and its deformation. We also restrict our argument to the case of $\gamma' = 0$, $\delta = \delta'^{-1} = \nu/\mu$ in (18) for simplicity and define (figure 2(b))

$$\begin{aligned} &f^{\mathrm{GLM}}(l,0,0) = f^{\mathrm{GLM}}(l,0,1) = f^{\mathrm{GLM}}(l,1,-1) = f^{\mathrm{GLM}}(l,1,1) =: z_l \\ &f^{\mathrm{GLM}}(l,1,0) = \frac{\mu-1}{\nu} \end{aligned} \tag{19}$$

where μ is a constant.

The dynamics of this model is described by the map

$$z_{l+1} = f(z_l) := \mu \frac{z_l(1-\gamma z_l)}{1+\nu(1-\gamma)z_l} \qquad z_l \in \mathbb{C}, l \in \mathbb{Z}. \tag{20}$$

We call this map a generalized logistic map (GLM). Some properties are listed below.

(1) When $\gamma = 1$, the map becomes the ordinary logistic map studied extensively in the literature.

(2) The GLM becomes the logistic equation for all values of the parameters γ, μ, ν when the continuous limit of the variable l is taken. To show this let us introduce new variables u and new parameters a and h,

$$u(l) := \frac{\nu+\gamma\mu-\gamma\nu}{\mu-1} z_l \qquad ah := \mu - 1. \tag{21}$$

We replace z_{l+1} by z_{l+h} and take the limit $h \to 0$. We will find that (20) reduces to the logistic equation

$$\frac{\mathrm{d}u}{\mathrm{d}l} = au(1-u). \tag{22}$$

(3) The GLM includes (10) as the special case with $\gamma = 0$. This explains the name of the ILM used for (10).

(4) The GLM generates a Julia set as long as $\gamma \neq 0$. Hence it is not integrable except for $\gamma = 0$. This will be discussed later.

The most important feature of the GLM is that it interpolates a non-integrable map to an integrable map in the limit of continuous deformation. This fact enables us to study analytically the transition between the two phases. The problem we are concerned with in what follows is the analytical properties of the map (20). To proceed further it is more convenient to convert the map (20) into the standard form of the rational map of degree 2,

$$F(z) = \phi \circ f \circ \phi^{-1}(z) = \frac{z(z+\lambda)}{1+\lambda' z} \,\mathrm{e}^{\mathrm{i}\theta} \tag{23}$$

by the Möbius transformation

$$\phi(x) = \frac{(1-\mu)x}{(\nu\gamma-\nu-\mu\gamma)x + (\nu\gamma-\nu-\gamma)\mu\,\mathrm{e}^{-\mathrm{i}\theta}} \tag{24}$$

where

$$\lambda = \mu\,\mathrm{e}^{-\mathrm{i}\theta} \qquad \lambda' = \frac{\nu\gamma-\nu-2\mu\gamma+\mu^2\gamma}{(\nu\gamma-\nu-\gamma)\mu}\,\mathrm{e}^{\mathrm{i}\theta}. \tag{25}$$

The corresponding integrable map turns out to be the following case:

$$F(z) = \mu z = \lambda\,\mathrm{e}^{\mathrm{i}\theta} z \qquad \text{if } \gamma = 0, \text{i.e. } \lambda\lambda' = 1. \tag{26}$$

The main feature of a dynamical system is determined by the nature of fixed points of the map. Namely, the multiplier Λ at a fixed point a of the map $\varphi(z)$ is defined by the derivative of the map at a:

$$\Lambda := \left.\frac{\mathrm{d}\varphi(z)}{\mathrm{d}z}\right|_{z=a}. \tag{27}$$

The fixed point a is an attractor of the map if $|\Lambda| < 1$, a repeller if $|\Lambda| > 1$, and neutral if $|\Lambda| = 1$.

In the case of the GLM, the fixed points are easily found as

$$0 \qquad p = -\lambda - \frac{1-\lambda\lambda'}{\lambda' - \mathrm{e}^{\mathrm{i}\theta}} \qquad \infty. \tag{28}$$

The corresponding multipliers are

$$\Lambda_0 = \lambda\,\mathrm{e}^{\mathrm{i}\theta} \qquad \Lambda_p = \frac{2-\lambda\,\mathrm{e}^{\mathrm{i}\theta} - \lambda'\,\mathrm{e}^{-\mathrm{i}\theta}}{1-\lambda\lambda'} \qquad \Lambda_\infty = \lambda'\,\mathrm{e}^{-\mathrm{i}\theta}. \tag{29}$$

In the integrable limit $\lambda\lambda' \to 1$, we observe the following characteristic features.

(1) Since

$$|\Lambda_0\Lambda_\infty| \to 1 \tag{30}$$

the map converges either to 0 or to ∞ depending on $|\lambda| = |\mu| < 1$ or >1.

(2) The fixed point p approaches $-\lambda$ and it turns to a superrepeller

$$|\Lambda_p| \to \infty. \tag{31}$$

4. Julia sets

In the complex dynamical systems, chaos appears from a Julia set. Given the map $f(z)$ on a Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the Riemann sphere is divided into two parts depending on whether the orbits converge or not. A set of initial values whose orbits, together with their neighbourhood, converge is called a Fatou set $F(f)$. On the other hand, a set which does not is called a Julia set $J(f)$. This definition leads to the fact that the Julia set does not contain any attractive periodic cycle. In this sense the orbit in the Julia set is chaotic.

By definition, a Fatou set and a Julia set are invariant with respect to the map, that is

$$f(F) = f^{-1}(F) = F \qquad f(J) = f^{-1}(J) = J.$$

It is easy to understand that attractive fixed points belong to the Fatou set. In contrast, it is known that repulsive fixed points belong to the Julia set [12]. Then we can compute the Julia set by inversely mapping a repulsive fixed point as an initial value. We show some of their examples in figure 3 for the map of (23).

The Julia set does not exist if the map is completely integrable. Integrable maps converge to orbits which are predictable for any given initial values. Conversely, if there exists an orbit which is not predictable for some initial values, the map is not integrable. Therefore, a Julia set appears in non-integrable maps, but not in integrable maps.

In our standard map of degree 2 given by (23), a Julia set is known to exist except for at the integrable point $\lambda\lambda' = 1$. We would like to know how it disappears from the complex plane of the variable when the parameters approach the limit $\lambda\lambda' \to 1$. We have given in [13] an argument about this problem for some limited range of parameters. The purpose of this section is to present another argument which should supplement our previous one.

The inverse map of (23) is easily obtained as

$$z_l = F^{-1}(z_{l+1}) = \tfrac{1}{2}(\rho z_{l+1} - \lambda) \pm \tfrac{1}{2}\sqrt{(\rho z_{l+1} + \lambda)^2 + 4z_{l+1}\,\mathrm{e}^{-\mathrm{i}\theta}(1-\lambda\lambda')} \tag{32}$$

where we have defined

$$\rho := \lambda' \,\mathrm{e}^{-\mathrm{i}\theta}. \tag{33}$$

From this expression it is apparent that the inverse map is not unique but double valued at every step. As we pointed out above, the inverse map generates points of the Julia set if it starts from a point on the Julia set. Substituting one value of the Julia set into (32), we get two points every time. After n steps the number of points of the Julia set increases by as many as $2^{n+1} - 1$. This explains the nature of the Julia set. Some of the points could be those of periodic maps. They must be subtracted from the number.

In the integrable limit $\lambda\lambda' \to 1$ the inverse map (32) is still double valued. The values are

$$z_l = \begin{cases} \rho z_{l+1} \\ -\lambda. \end{cases} \tag{34}$$

We note that the second solution does not depend on z_{l+1}, hence is the same at every step of the map. For (34) to generate the Julia set we must start from a repulsive fixed point. When $|\lambda| > 1$, the origin is such a point. Thence we find from (34) the 'Julia set'†

$$J^{\mathrm{ILM}} = \{-\rho^n \lambda \mid n \in \mathbb{N}\} \tag{35}$$

for the integrable map. The number of the 'Julia set' increases proportionally to the number of steps n. Moreover, the element of J^{ILM} is equal to $-\mu^{-n}\lambda$, which is nothing but the solution exactly expected from the map (26), if it started from $-\lambda$.

The next problem we are concerned with is to explore how the Julia set of the GLM turns into those points of (34) in the limit $\lambda\lambda' \to 1$. Since we are interested in the transition from a non-integrable map to the integrable map, we are to consider small values of $|\lambda\lambda' - 1|$. The inverse map (32) can be rewritten as

$$F^{-1}(z) = \left\{ \begin{matrix} \rho z \\ -\lambda \end{matrix} \right\} \pm E(z) \tag{36}$$

where we put

$$E(z) := \frac{1}{2}(\rho z + \lambda)\left(\sqrt{1 - \frac{4z\epsilon\,\mathrm{e}^{-\mathrm{i}\theta}}{(\rho z + \lambda)^2}} - 1\right) \qquad \epsilon := \lambda\lambda' - 1. \tag{37}$$

Note that $E(z)$ vanishes for small values of ϵ. To see the behaviour of $E(z)$ for small ϵ we first observe the inequality which is true for all ϵ:

$$|E(z)| < 3\sqrt{|z\epsilon|} \qquad \forall \epsilon \in \mathbb{C}. \tag{38}$$

The proof of this inequality can be found with the following facts.

(1) If $|w| < 1$,

$$\begin{aligned} \left|\frac{1}{\sqrt{w}}(\sqrt{1-w} - 1)\right| &= \frac{1}{|\sqrt{w}|}|1 - \sqrt{1-w}| \leqslant \frac{1}{|\sqrt{w}|}(1 - \sqrt{1 - |w|}) \\ &\leqslant \frac{1}{|\sqrt{w}|}(1 - (1 - |w|)) = |\sqrt{w}| \leqslant 1. \end{aligned} \tag{39}$$

(2) If $|w| > 1$,

$$\left|\frac{1}{\sqrt{w}}(\sqrt{1-w} - 1)\right| = \left|\sqrt{\frac{1}{w} - 1} - \sqrt{\frac{1}{w}}\right| < 3. \tag{40}$$

† This set does not possess properties expected for the ordinary Julia set. We call it a 'Julia set' only in the sense that it is generated by the inverse map starting from a repeller.

Substituting

$$w := \frac{4z\epsilon\,\mathrm{e}^{-\mathrm{i}\theta}}{(\rho z+\lambda)^2} \tag{41}$$

into $E(z)$ of (37), we can write

$$|E(z)| = \sqrt{|z\epsilon|}\left|\frac{1}{\sqrt{w}}(\sqrt{1-w}-1)\right| \tag{42}$$

from which (38) follows.

We can perform the inverse map (38) iteratively. Let us denote the map (36) as

$$A(z) := \rho z + E(z) \qquad B(z) := -\lambda - E(z). \tag{43}$$

Then the second map becomes

$$F^{-2}(z) = \begin{cases} A(F^{-1}(z)) \\ B(F^{-1}(z)) \end{cases} = \begin{cases} A\circ A(z) \\ A\circ B(z) \\ B\circ A(z) \\ B\circ B(z). \end{cases} \tag{44}$$

After n steps we obtain

$$F^{-n}(z) = \{A^{\nu_1}\circ B^{\nu_2}\circ A^{\nu_3}\circ\cdots\circ B^{\nu_n}(z)|\nu_1+\nu_2+\cdots+\nu_n = n\}. \tag{45}$$

If we had started from the repeller the above maps would have produced the Julia set of the GLM. In the following we consider the case $|\lambda|>1$, $|\lambda'|<1$, so that the origin is a repulsive fixed point and the infinity is an attractive fixed point. Since $E(0)=0$ the origin is mapped to

$$A(0) = 0 \qquad B(0) = -\lambda \tag{46}$$

by the first iteration. The second iteration yields

$$\begin{array}{ll} A^2(0) = 0 & A\circ B(0) = -\rho\lambda + E(-\lambda) \\ B\circ A(0) = -\lambda & B^2(0) = -\lambda - E(-\lambda). \end{array} \tag{47}$$

We note that since $E(-\lambda)$ is the order of ϵ from (38), all of the points after the second iteration are in the neighbourhood of J^{ILM}. Proceeding similarly, we obtain the Julia set as follows:

$$J^{\mathrm{GLM}} = \{A^{\nu_1}\circ B^{\nu_2}\circ A^{\nu_3}\circ\cdots\circ B^{\nu_\infty}(0)|\nu_1,\nu_2,\ldots\in\mathbb{N}\}. \tag{48}$$

We note some important properties which result from this expression.

(1) The invariance of the Julia set under the map: It is obvious from (48) that

$$J^{\mathrm{GLM}} = A(J^{\mathrm{GLM}})\cup B(J^{\mathrm{GLM}}) = F^{-1}(J^{\mathrm{GLM}}). \tag{49}$$

(2) An element of the form $B\circ X$ for any $X\in J^{\mathrm{GLM}}$ belongs to the neighbourhood of $-\lambda$, as seen from

$$B\circ X = -\lambda - E(X). \tag{50}$$

(3) An element of the form $A^s\circ B\circ X$ maps $B\circ X$ to the neighbourhood of $-\rho^s\lambda$. In fact after applying A s times we get

$$\begin{aligned} A^s(B\circ X) &= A^{s-1}(\rho B\circ X + E(B\circ X)) \\ &= A^{s-2}(\rho^2 B\circ X + \rho E(B\circ X) + E(A\circ B\circ X)) \\ &= \rho^s B\circ X + \sum_{k=0}^{s-1}\rho^k E(A^{s-k-1}\circ B\circ X) \\ &= -\rho^s\lambda - \rho^s E(X) + \sum_{k=0}^{s-1}\rho^k E(A^{s-k-1}\circ B\circ X). \end{aligned} \tag{51}$$

Since every element of J^{GLM}, beside 0, is either in the form of $B\circ X$ or $A^s\circ B\circ X$, we conclude that every element of J^{GLM} is in the neighbourhood of J^{ILM}.

We now proceed to show that J^{GLM} approaches uniformly to J^{ILM} as ϵ goes to 0. Since the infinity is an attractive fixed point, the Julia set must be in a finite region of the complex plane. We assume that they are inside of the disc of radius R, i.e., $|z| < R$, $\forall z \in J^{\mathrm{GLM}}$. Therefore, we can bound $|E(z)|$ by

$$|E(z)| < 3\sqrt{R}\sqrt{|\epsilon|} \qquad z\in J^{\mathrm{GLM}}. \tag{52}$$

The summation of (51) can be estimated as

$$\sum_{k=0}^{s-1}|\rho^k E(A^{s-k-1}BX)| < \left(\sum_{k=0}^{s-1}|\rho|^k\right)3\sqrt{R}\sqrt{|\epsilon|} = \frac{1-|\rho|^s}{1-|\rho|}3\sqrt{R}\sqrt{|\epsilon|} \tag{53}$$

which vanishes as $|\epsilon|$ approaches 0 for all integers s because we assume $|\rho|^s = |\lambda'|^s < 1$. This proves that all points in J^{GLM} approach to J^{ILM} uniformly in the integrable limit.

In the above we have considered the case of $|\lambda| > 1, |\lambda'| < 1$. The other case $|\lambda| > 1, |\lambda'| < 1$ can be also treated similarly if z_l is transformed into $w_l = 1/z_l$ in (23). Since this transformation is equivalent to the exchange of the role of λ and λ' (and replacement of θ by $-\theta$) in (23), we can replay on the w-plane the same argument to the above.

We conclude this paper by showing pictures which represent the convergence of the Julia set to the points of iterative maps of the integrable system. The parameters of the map are fixed at $\lambda = 4$ and $\theta = 0.03\pi$. Under the choice of these parameters, λ', and hence ϵ, can be changed freely. In the integrable limit $\epsilon = 0$, $J^{\mathrm{ILM}} = \{-4(1/4^n)\,\mathrm{e}^{-\mathrm{i}0.03\pi n}, n = 0, 1, 2, \ldots\}$. It is shown in figure 3(c). As ϵ differs from 0 the Julia set expands from these points as seen in figures 3(b) and (a). The real and imaginary axes are not drawn except in figure 3(a), so that the points in the neighbourhoods of $z = -4$ and 0 are visible in the other figures.

By studying the analytical property of a piece of Toda lattice we have attempted to clarify how a non-integrable system approaches an integrable one. Our argument is based on the fact that the two-dimensional Toda lattice can be disjoined into small pieces, which are integrable by themselves and are called Toda molecules. A Toda molecule is composed from smaller pieces, which we called Toda atoms. Hence the two-dimensional Toda lattice is a crystal consisting of Toda atoms. For such a macroscopic system to be integrable every piece must be joined very carefully not to create a Julia set.

In this connection it is worthwhile recalling that a similar property is possessed commonly by other integrable models. In the solvable lattice models the partition function is factorizable into a product of Boltzmann weights. The Yang–Baxter equation is a condition imposed on the factors to be connected properly. Another example is the factorizability condition imposed on the string amplitudes which led us to the τ function of the KP hierarchy. In any case, the connection rule must be such that the symmetry characterizing the unit blocks is preserved under the coupling.

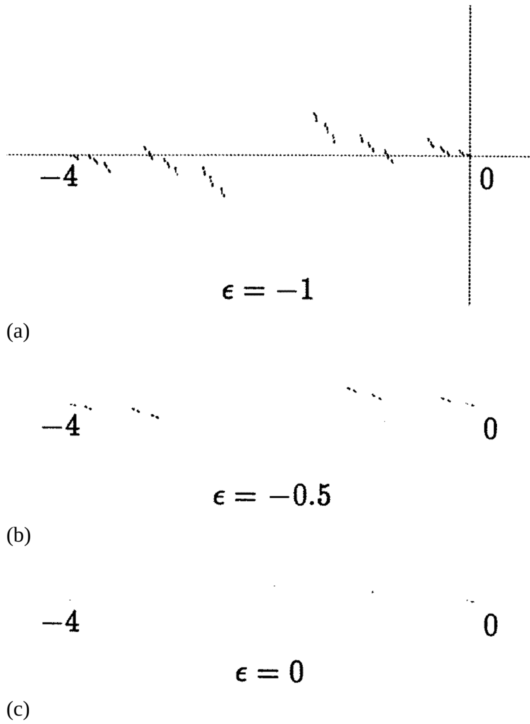

Figure 3.

We will be interested in studying analytically properties of the compound system of two GLM pieces in a forthcoming paper.

Acknowledgments

This work was supported in part by the Grant-in-Aid for general Scientific Research from the Ministry of Education, Science, Sports and Culture, Japan (No 06835023), and the Fiscal Year 1996 Fund for Special Research Projects at Tokyo Metropolitan University.

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